JOURNAL OF COMPUTATIONAL PHYSICS 12, 137–142 (1973)

Note

Calculation of the Capacitance of a Ring Capacitor of Rectangular Cross Section with no Insulating Gaps

I. Introduction

Ring capacitors exhibit characteristics similar to the Thompson–Lampard four cylinder capacitors [1, 2] presently the international standard of capacitance. They have the property that fringe effects of the ends, which are difficult to account for in the Thompson–Lampard capacitor, are nonexistent in a ring structure. Studies have been in progress at National Research Council [3, 4, 5] with the aim of finding the best cross section for use as a standard capacitor and as the main element in a device to measure the angle of arc. In this note, the case of a rectangular cross section with no insulating gaps, shown in Fig. 1, is described.

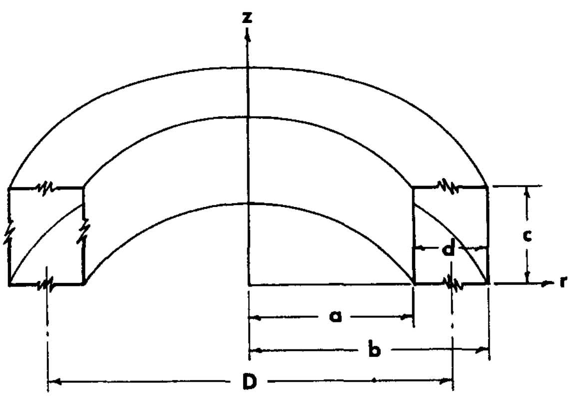

Fig. 1. A section of a ring capacitor of rectangular cross section with no insulating gaps.

II. Series Solution for the Capacitance

The series solutions for the capacitances were obtained from D. G. Lampard [6]. Further simplifications of these solutions were made.

The solution of the problem is obtained in circular cylindrical coordinates (r, ϕ, z). The inner and outer walls of the capacitor are given by $r = a$ and $r = b$, respectively, and the upper and lower walls of the capacitor are given by $z = 0$ and $z = c$, respectively.

When the outer wall is at unit potential and all other walls are at zero potential, the potential $V_1(r, z)$ at a point (r, ϕ, z) inside the capacitor is given by [7, Section 5.36]

$$V_1(r, z) = \sum_{\substack{n\,\mathrm{odd}\\ n>0}} \frac{4}{n\pi} \frac{[I_0(n\pi r/c)/I_0(n\pi a/c)] - [K_0(n\pi r/c)/K_0(n\pi a/c)]}{[I_0(n\pi b/c)/I_0(n\pi a/c)] - [K_0(n\pi b/c)/K_0(n\pi a/c)]} \sin(n\pi z/c), \quad (1)$$

where $I_0(x)$ and $K_0(x)$ are modified Bessel functions of the first and second kind, respectively, of order 0.

The direct capacitance C_1 between the inner and outer walls of the capacitor is then obtained from the expression

$$C_1 = a/2 \int_0^c \partial V_1/\partial r \Big|_{r=a} dz.$$

This gives the result

$$C_1 = \frac{4c}{\pi^2} \sum_{\substack{n\,\mathrm{odd}\\ n>0}} \frac{1}{n^2[K_0(n\pi a/c)\, I_0(n\pi b/c) - I_0(n\pi a/c)\, K_0(n\pi b/c)]}\,. \quad (2)$$

When the upper wall is at unit potential and all other walls are at zero potential, the potential $V_2(r, z)$ at a point (r, ϕ, z) inside the capacitor is given by [7, Section 5.304],

$$V_2(r, z) = \sum_{n=1}^{\infty} \frac{2}{\mu_n} \frac{\sinh(\mu_n z)}{\sinh(\mu_n c)} \frac{-R_0(\mu_n r)}{bR_0'(\mu_n b) + aR_0'(\mu_n a)}\,, \quad (3)$$

where

$$R_0(\mu_n r) = J_0(\mu_n r)\, Y_0(\mu_n b) - J_0(\mu_n b)\, Y_0(\mu_n r), \quad (4)$$

$$R_0' = \frac{1}{\mu_n} \frac{dR_0}{dr} = -J_1(\mu_n r)\, Y_0(\mu_n b) + J_0(\mu_n b)\, Y_1(\mu_n r). \quad (5)$$

μ_n is the n-th positive root of the equation

$$R_0(\mu a) \equiv J_0(\mu a)\, Y_0(\mu b) - J_0(\mu b)\, Y_0(\mu a) = 0, \quad (6)$$

and $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind, respectively, of order 0.

The direct capacitance C_2 between the upper and lower walls of the capacitor is then obtained from the expression,

$$C_2 = \tfrac{1}{2}\int_a^b r\, \partial V_2/\partial z\Big|_{z=0} dr,$$

and is given by

$$C_2 = \sum_{n=1}^{\infty} \frac{1}{\mu_n \sinh(\mu_n c)} \frac{bR_0'(\mu_n b) - aR_0'(\mu_n a)}{bR_0'(\mu_n b) + aR_0'(\mu_n a)}. \tag{7}$$

The Wronskian relationship for Bessel functions gives

$$J_1(x)\, Y_0(x) - J_0(x)\, Y_1(x) = 2/(\pi x). \tag{8}$$

If $J_0(\mu_n a) \neq 0$, then solving (6) for $Y_0(\mu_n b)$, substituting in (5), and using (8), we get

$$R_0'(\mu_n a) = -[2J_0(\mu_n b)]/[\pi\mu_n a J_0(\mu_n a)]. \tag{9}$$

From (8), it is seen that

$$R_0'(\mu_n b) = -2/(\pi\mu_n b). \tag{10}$$

From (7), (9), and (10), we get

$$C_2 = \sum_{n=1}^{\infty} \frac{1}{\mu_n \sinh(\mu_n c)} \frac{J_0(\mu_n a) - J_0(\mu_n b)}{J_0(\mu_n a) + J_0(\mu_n b)}. \tag{11}$$

If $J_0(\mu_n a) = 0$, then $Y_0(\mu_n a) \neq 0$, and the term,

$$[J_0(\mu_n a) - J_0(\mu_n b)]/[J_0(\mu_n a) + J_0(\mu_n b)],$$

in (11) should be replaced by

$$[Y_0(\mu_n a) - Y_0(\mu_n b)]/[Y_0(\mu_n a) + Y_0(\mu_n b)].$$

III. Weights

In order that changes in the dimension c of the capacitor have a second-order effect on the capacitance, weights [8] are used. They are defined as follows:

$$w_1 = \frac{-\partial C_2/\partial c}{(\partial C_1/\partial c) - (\partial C_2/\partial c)} \quad \text{and} \quad w_2 = \frac{\partial C_1/\partial c}{(\partial C_1/\partial c) - (\partial C_2/\partial c)}. \tag{12}$$

The capacitance of the capacitor is defined as

$$C = w_1 C_1 + w_2 C_2 . \tag{13}$$

We see from (12) that

$$w_1 + w_2 = 1. \tag{14}$$

It follows from the maximum principle for solutions to elliptic equations that

$$\partial C_1/\partial c > 0 \qquad \text{and} \qquad \partial C_2/\partial c < 0.$$

Therefore

$$w_1 > 0 \qquad \text{and} \qquad w_2 > 0.$$

Suppose the weights deviate from those defined by (12) for the dimensions a, b, and c, but still satisfy (14) so that the weights used are $\tilde{w}_1 = w_1 + \delta$ and $\tilde{w}_2 = w_2 - \delta$ and let the defined value of capacitance, $\tilde{C}$, be given by the equation

$$\tilde{C} = \tilde{w}_1 C_1(a, b, c) + \tilde{w}_2 C_2(a, b, c).$$

Then, keeping w_1 and w_2 fixed, the change $\Delta\tilde{C}$ in $\tilde{C}$ due to a change ϵ in the dimension c is given by

$$\Delta\tilde{C} = (\partial C_1/\partial c - \partial C_2/\partial c)\, \delta\epsilon + o(\epsilon^2).$$

IV. Computed Results

A detailed description of computational techniques is given in [9]. An IBM 360, Model 67 computer was used.

Given values a and b, Table I gives the value $c = c_0$ for which the two capacitances are equal, the weight w_1, and the value of capacitance for the values a, b, and c_0. The Chebyshev series were computed as the limit as $n \to \infty$ of the Chebyshev interpolation polynomials of degree n [10]. The coefficients of the Chebyshev interpolation polynomials of degree 10 were used for Table I. The value c for which the capacitances are equal was determined by Newton–Raphson iteration on c.

The Bessel functions $J_n(x)$ and $I_n(x)$ were generated by backward recurrence techniques, the Bessel functions $Y_n(x)$ were generated from Chebyshev expansions, and the Bessel functions $K_n(x)$ were generated from their power series by a continued fraction technique or from their asymptotic expansions depending on the value of the argument. The subroutines used to generate $I_n(x)$ and $K_n(x)$ return

TABLE I

$$D = b + a,\ d = b - a$$
$$x = 8(d/D)^2 - 1,\ 0 < (d/D) < 1/2$$
$$C_1 = C_2 = C_0 \text{ for } c = c_0$$
$$c_0/d = \sum_{i=0}^{\infty} a_i T_i(x),\ w_1 = \sum_{i=0}^{\infty} b_i T_i(x)$$
$$C_0/(\pi D) = \sum_{i=0}^{\infty} e_i T_i(x)$$

i	a_i	b_i	$e_i \times 100$
0	1.01004 41311	0.50260 80861	1.71742 17771
1	0.01039 55828	0.00268 51523	−0.03933 35907
2	0.00036 75721	0.00008 04275	−0.00103 36033
3	0.00001 69554	0.00000 35321	−0.00004 25284
4	0.00000 08814	0.00000 01800	−0.00000 20789
5	0.00000 00491	0.00000 00099	−0.00000 01113
6	0.00000 00029	0.00000 00006	−0.00000 00063
7	0.00000 00002	0.00000 00000	−0.00000 00004

Note: The value $C_0/(\pi D)$ is expressed in esu/cm. For $d/D = 0$, it assumes the value $\log 2/(4\pi^2)$.

the values $e^{-x}I_n(x)$ and $e^{x}K_n(x)$. This prevents number overflow and underflow. $K_m(x)\,I_n(y)$ was computed as the product of e^{-x+y}, $e^{x}K_m(x)$, and $e^{-y}I_n(y)$.

The roots of Eq. (6) were computed from their asymptotic expansion if this is sufficiently accurate. If not, Newton–Raphson iteration is used and a sufficiently large sequence of roots is computed so that a root obtained by Newton–Raphson iteration can be compared to its value obtained from the asymptotic expansion to check the correctness of the sequence of roots. The method of false position was used to obtain a first estimate for Newton–Raphson iteration. If the asymptotic expansion for a root gives a closer estimate than the method of false position, then the asymptotic expansion is used to obtain a first estimate of all succeeding roots computed by Newton–Raphson iteration.

VII. Conclusion

The ring capacitor exhibits properties which are of interest in capacitance standards. The second order dependence on dimensional perturbations is of importance in the construction and assembly stage and promise good subsequent stability. A practical embodiment of this ring capacitor and other types is described in [5].

REFERENCES

1. D. G. LAMPARD, A new theorem in electrostatics with applications to calculable standards of capacitance, *Proc. I.E.E.*, Monograph No. 216M (Jan., 1957) 104C, p. 271.
2. A. M. THOMPSON, The cylindrical cross-capacitor as a calculable standard, *Proc. I.E.E.*, Paper No. 2887M (May, 1959) 106B, p. 307.
3. D. MAKOW, A new computable capacitor, *Metrologia*, **5** (1969), 126–128.
4. D. MAKOW, A new stable capacitor and its application to the measurement of angle, *Radio Sci.* **6** (1971), 1–5.
5. D. MAKOW AND J. B. CAMPBELL, Circular four electrode capacitors for capacitance standards, *Metrologia* **8** (1972), 148–155.
6. D. G. LAMPARD, unpublished material, private communication.
7. W. R. SMYTHE, "Static and Dynamic Electricity," McGraw-Hill Book Company, Inc., New York, 1968.
8. J. Q. SHIELDS AND R. D. CUTKOSKY, unpublished material, private communication.
9. J. B. CAMPBELL AND D. MAKOW, Numerical analysis of a ring capacitor of rectangular cross section with no insulating gaps, *Res. Rep.*, in print.
10. F. B. HILDEBRAND, "Introduction to Numerical Analysis," pp. 389–391. McGraw-Hill Book Company, Inc., New York, 1956.

RECEIVED: May 18, 1972

J. B. CAMPBELL

Computation Centre,
National Research Council,
Ottawa, Canada

AND

D. MAKOW

Division of Physics,
National Research Council,
Ottawa, Canada